

Homogenization error for two scale Maxwell equations ^{*}

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Abstract

For two scale elliptic equations in a domain D , standard homogenization errors are deduced with the assumption that the solution u_0 of the homogenized equation belongs to $H^2(D)$. For two scale Maxwell equations, the corresponding required regularity is $u_0 \in H^1(\text{curl}, D)$. These regularity conditions normally do not hold in general polygonal domains, which are of interests for finite element discretization. The paper establishes homogenization errors when u_0 belongs to a weaker regularity space $H^{1+s}(D)$ for elliptic problems and $H^s(\text{curl}, D)$ for Maxwell problems where $0 < s < 1$. Though we only present the results for two scale Maxwell equations when $u_0 \in H^s(\text{curl}, D)$ with $0 < s < 1$, the procedure works verbatim for elliptic equations when u_0 belongs to $H^{1+s}(D)$ with $0 < s < 1$.

1 Introduction

For two scale elliptic problems in a domain $D \subset \mathbb{R}^d$, standard homogenization error estimates are deduced with the assumption that the solution u_0 of the homogenized equation belongs to $H^2(D)$ (see, e.g., [2] and [6]). In many cases, this condition does not hold. A typical situation is that of nonconvex polygonal domains which we often need to consider in the context of finite element discretization. For Maxwell equations, the approaches in [2], [6] and [3] require the regularity $u_0 \in H^1(\text{curl}, D)$ (see Section 3) which normally does not hold in polygonal domains. However, in polygonal domains, the solutions u_0 belong to a weaker regularity space, i.e $u_0 \in H^{1+s}(D)$ for elliptic problems and $u_0 \in H^s(\text{curl}, D)$ for Maxwell problems with $0 < s < 1$ (see, e.g., [4]). The paper develops new homogenization errors for two scale Maxwell problems where $u_0 \in H^s(\text{curl}, D)$ for $0 < s < 1$. Though we only present the results for two scale Maxwell equations, our approach applies verbatim for two scale elliptic equations with $u_0 \in H^{1+s}(D)$ for $0 < s < 1$.

The paper is organized as follows. In the next section, we introduce the two scale Maxwell equation (2.3) and consider its homogenization. Homogenization limit of Maxwell equation (2.3) is deduced in Benssousan et al. [2] though the two scale asymptotic expansion is performed only for the case where the coefficient b^ε in (2.3) is a constant isotropic matrix; and as a consequence the corrector function u_1 in (2.8) is not derived. To get the correctors explicitly, we therefore employ the two scale convergence method ([8] and [1]) to establish the two scale homogenized equation, from which we deduce explicit formulae for the two correctors u_1 and u_1 in (2.11) and (2.8) respectively. We note that two scale convergence limits of bounded sequences in $H(\text{curl}, D)$ are considered in [9] but the form of the limit function is slightly different from the form that we need in this paper. We therefore derive and prove in details the two scale convergence limits. Section 3 establishes the homogenization error when $u_0 \in H^1(\text{curl}, D)$. We follow the standard approach of [6] for elliptic equations, but it appears that this has not been done for Maxwell equations in the literature. Our main contribution is contained in Section 4 where we derive homogenization errors for the case where $u_0 \in H^s(\text{curl}, D)$ for $0 < s < 1$. The last section proves the regularity required for the solutions of the cell problems (2.7) and (2.10), and for u_0 .

Throughout the paper, by ∇ and curl , without indicating explicitly the variable, we denote the total gradient and curl of a function of the variable x . Partial gradient and partial curl of a function that depends on x and y are denoted by ∇_x , ∇_y , curl_x and curl_y respectively. Repeated indices indicate summation. The notation $\#$ denotes spaces of periodic functions; c denotes various constants of different values.

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2 Homogenization of the Maxwell equation

Let D be a domain in \mathbb{R}^3 with a Lipschitz boundary. Let Y be the open unit cube in \mathbb{R}^3 . Let $a(x, y), b(x, y) : D \times Y \rightarrow \mathbb{R}^{3 \times 3}$ be positive definite matrix functions, i.e. there are constants $c_1, c_2 > 0$ so that for all vectors $\xi, \zeta \in \mathbb{R}^3$ we have

$$c_1 |\xi|^2 \leq a(x, y) \xi \cdot \xi, \quad c_1 |\zeta|^2 \leq b(x, y) \zeta \cdot \zeta, \quad (2.1)$$

and

$$a(x, y) \xi \cdot \zeta \leq c_2 |\xi| |\zeta|, \quad b(x, y) \xi \cdot \zeta \leq c_2 |\xi| |\zeta| \quad (2.2)$$

for all $x \in D$ and $y \in Y$, where $|\cdot|$ denotes the Euclid norm in \mathbb{R}^3 . We define the multiscale coefficients $a^\varepsilon : D \rightarrow \mathbb{R}^{3 \times 3}$ and $b^\varepsilon : D \rightarrow \mathbb{R}^{3 \times 3}$ as

$$a^\varepsilon(x) = a\left(x, \frac{x}{\varepsilon}\right), \quad b^\varepsilon(x) = b\left(x, \frac{x}{\varepsilon}\right).$$

Let $V = H_0(\text{curl}, D)$ and $f \in V'$. We consider the problem

$$\text{curl}(a^\varepsilon(x) \text{curl } u^\varepsilon) + b^\varepsilon u^\varepsilon = f, \quad u^\varepsilon \times n = 0, \quad \text{on } \partial D, \quad (2.3)$$

where n is the outward normal vector on ∂D . In variational form, this problem is written as

$$\int_D [a^\varepsilon(x) \text{curl } u^\varepsilon \cdot \text{curl } \phi + b^\varepsilon u^\varepsilon \cdot \phi] dx = \int_D f \cdot \phi dx \quad \forall \phi \in V. \quad (2.4)$$

We study homogenization of equation (2.3) via two scale convergence in this section. We first recall the concept of two scale convergence (see [8] and [1]).

Definition 2.1 A sequence of functions $\{w^\varepsilon\}_\varepsilon \subset L^2(D)$ two scale converges to a function $w_0 \in L^2(D \times Y)$ if for all smooth functions $\phi \in C^\infty(D \times Y)$ which are periodic with respect to y with the period being Y

$$\lim_{\varepsilon \rightarrow 0} \int_D w^\varepsilon(x) \phi(x, \frac{x}{\varepsilon}) dx = \int_D \int_Y w_0(x, y) \phi(x, y) dy dx.$$

The definition makes sense due to the following result.

Proposition 2.2 From a bounded sequence in $L^2(D)$ we can extract a two scale convergent subsequence.

To deduce the homogenized problem for the two scale Maxwell equation (2.3) by two scale convergence, we first establish two scale convergence results for a bounded sequence in $H(\text{curl}, D)$. These results are first established in Wellander and Kristensson [9]. However, for completeness we present the proof here as the convergence limit we use is in a slightly different form from that of [9] (see Remark 2.4).

Let $H_\#(\text{curl}, Y)$ be the space of functions in $H(\text{curl}, Y)$ which can be extended periodically to $H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$. This space is the closure of the space $C_\#^\infty(Y)$ of periodic smooth functions in $H(\text{curl}, Y)$. We denote by $\tilde{H}_\#(\text{curl}, Y)$ the space of equivalent classes of functions in $H_\#(\text{curl}, Y)$ where ϕ and ψ are regarded as equal if $\text{curl}_y \phi = \text{curl}_y \psi$, with the norm $\|\phi\|_{\tilde{H}_\#(\text{curl}, Y)} = \|\text{curl}_y \phi\|_{L^2(Y)^3}$.

Proposition 2.3 Let $\{w^\varepsilon\}_\varepsilon \subset H(\text{curl}, D)$. There is a subsequence (not renumbered), and functions $w_0 \in L^2(D)^3$, $w_1 \in L^2(D, H_\#^1(Y)/\mathbb{R})$, $w_1 \in L^2(D, \tilde{H}_\#(\text{curl}, Y))$ such that

$$w^\varepsilon \xrightarrow{\text{two scale}} w_0 + \nabla_y w_1, \quad \text{and} \quad \text{curl } w^\varepsilon \xrightarrow{\text{two scale}} \text{curl } w_0 + \text{curl}_y w_1.$$

Proof As $\{w^\varepsilon\}_\varepsilon$ is bounded in $H(\text{curl}, D)$, there is a subsequence (not renumbered) such that w^ε and $\text{curl } w^\varepsilon$ are two scale convergent. Let $\xi \in L^2(D \times Y)^3$ be the two scale limit of $\{w^\varepsilon\}_\varepsilon$. Let $\phi(x, y) = \varepsilon \Phi(x, y)$ where Φ is a smooth function in $C_0^\infty(D, C_\#^\infty(Y))^3$. We have

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \int_D \text{curl } w^\varepsilon \cdot \varepsilon \Phi(x, \frac{x}{\varepsilon}) dx = \lim_{\varepsilon \rightarrow 0} \int_D w^\varepsilon \cdot \varepsilon \text{curl } \Phi(x, \frac{x}{\varepsilon}) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_D w^\varepsilon \cdot \text{curl}_y \Phi(x, \frac{x}{\varepsilon}) dx \\ &= \int_D \int_Y \xi(x, y) \cdot \text{curl}_y \Phi(x, y) dy dx. \end{aligned}$$

As this holds for all the smooth functions $\Phi(x, y) \in C_0^\infty(D, C_\#^\infty(Y))$, we have that

$$\xi(x, y) = \xi_0(x) + \nabla_y \mathbf{w}_1(x, y)$$

for a function $\mathbf{w}_1(x, y) \in L^2(D, H_\#^1(Y)/\mathbb{R})$ and a function $\xi_0 \in L^2(D)^3$. As $\int_Y \xi(x, y) dy = \xi_0(x)$, ξ_0 equals the weak limit w_0 of w^ε in $L^2(D)^3$.

Let $\eta(x, y) \in L^2(D \times Y)^3$ be the two scale limit of $\text{curl } w^\varepsilon$. For $\Phi \in C_0^\infty(D, C_\#^\infty(Y))$, we have

$$\int_D \text{curl } w^\varepsilon \cdot \nabla \Phi(x, \frac{x}{\varepsilon}) dx = \int_D w^\varepsilon \cdot \text{curl } \nabla \Phi(x, \frac{x}{\varepsilon}) dx - \int_{\partial D} (w^\varepsilon \times n) \cdot \nabla \Phi(x, \frac{x}{\varepsilon}) ds = 0.$$

Thus

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \int_D \text{curl } w^\varepsilon \cdot \varepsilon \nabla \Phi(x, \frac{x}{\varepsilon}) = \lim_{\varepsilon \rightarrow 0} \int_D \text{curl } w^\varepsilon \cdot \nabla_y \Phi(x, \frac{x}{\varepsilon}) dx \\ &= \int_D \int_Y \eta(x, y) \cdot \nabla_y \Phi(x, y) dy dx. \end{aligned}$$

Thus there is a function $w_1(x, y) \in L^2(D, \tilde{H}_\#(\text{curl}, Y))$ and a function $\eta_0(x) \in L^2(D)^3$ such that

$$\eta(x, y) = \eta_0(x) + \text{curl}_y w_1(x, y).$$

As $\int_Y \eta(x, y) dy = \eta_0(x)$, η_0 equals the weak limit of $\text{curl } w^\varepsilon$ in $L^2(D)^3$, which is $\text{curl } w_0$. \square

Remark 2.4 Wellander and Kristensson [9] show that if w^ε two scale converges to $w(x, y)$, then there is a function $\tilde{w}_1(x, y) \in L^2(D, H_\#(\text{curl}, D))$ so that the two scale limit $\eta(x, y)$ of $\text{curl } w^\varepsilon$ satisfies $\eta(x, y) = \text{curl}_x w(x, y) + \text{curl}_y \tilde{w}_1(x, y)$. As $w(x, y) = w_0(x) + \nabla_y \mathbf{w}_1(x, y)$, $\text{curl}_x w(x, y) = \text{curl } w_0(x) + \text{curl}_x \nabla_y \mathbf{w}_1(x, y) = \text{curl } w_0(x) - \text{curl}_y \nabla_x \mathbf{w}_1(x, y)$. Therefore

$$\eta(x, y) = \text{curl } w_0(x) + \text{curl}_y (-\nabla_x \mathbf{w}_1(x, y) + \tilde{w}_1(x, y)).$$

Our result above is consistent with the result by Wellander and Kristensson [9].

We now derive the two scale homogenized problem for the two scale Maxwell equation (2.3).

Proposition 2.5 There are functions $u_0(x) \in H_0(\text{curl}, D)$, $u_1(x, y) \in L^2(D, \tilde{H}_\#(\text{curl}, Y))$ and $\mathbf{u}_1(x, y) \in L^2(D, H_\#^1(Y)/\mathbb{R})$ such that

$$u^\varepsilon \xrightarrow{\text{two scale}} u_0 + \nabla_y \mathbf{u}_1,$$

and

$$\text{curl } u^\varepsilon \xrightarrow{\text{two scale}} \text{curl } u_0 + \text{curl}_y u_1.$$

The functions u_0 , u_1 and \mathbf{u}_1 satisfy the problem

$$\begin{aligned} \int_D \int_Y [a(x, y)(\text{curl } u_0 + \text{curl}_y u_1) \cdot (\text{curl } v_0 + \text{curl}_y v_1) + b(x, y)(u_0 + \nabla_y \mathbf{u}_1) \cdot (v_0 + \nabla_y \mathbf{v}_1)] dy dx \\ = \int_D f(x) \cdot v_0(x) dx \end{aligned} \quad (2.5)$$

for all $v_0 \in H_0(\text{curl}, D)$, $v_1 \in L^2(D, \tilde{H}_\#(\text{curl}, Y))$ and $\mathbf{v}_1 \in L^2(D, H_\#^1(Y)/\mathbb{R})$.

Proof The proof is standard. From (2.1) and (2.2), $\{u^\varepsilon\}_\varepsilon$ is uniformly bounded in $H(\text{curl}, D)$. Thus we can extract a subsequence (not renumbered) so that

$$u^\varepsilon \xrightarrow{\text{two scale}} u_0 + \nabla_y \mathbf{u}_1, \quad \text{and} \quad \text{curl } u^\varepsilon \xrightarrow{\text{two scale}} \text{curl } u_0 + \text{curl}_y u_1$$

for $u_0(x) \in H_0(\text{curl}, D)$, $u_1(x, y) \in L^2(D, \tilde{H}_\#(\text{curl}, Y))$ and $\mathbf{u}_1(x, y) \in L^2(D, H_\#^1(Y)/\mathbb{R})$. We choose functions $v_0 \in C_0^\infty(D)^3$, $v_1 \in C_0^\infty(D, C_\#^\infty(Y))^3$ and $\mathbf{v}_1 \in C_0^\infty(D, C_\#^\infty(Y))$. Let

$$\phi(x) = v_0(x) + \varepsilon v_1(x, \frac{x}{\varepsilon}) + \varepsilon \nabla \mathbf{v}_1(x, \frac{x}{\varepsilon}).$$

as the test function in (2.4). We get

$$\begin{aligned} & \int_D \left[a(x, \frac{x}{\varepsilon}) \operatorname{curl} u^\varepsilon(x) \cdot \left(\operatorname{curl} v_0(x) + \varepsilon \operatorname{curl}_x v_1(x, \frac{x}{\varepsilon}) + \operatorname{curl}_y v_1(x, \frac{x}{\varepsilon}) \right) \right. \\ & \quad \left. + b(x, \frac{x}{\varepsilon}) u^\varepsilon(x) \cdot \left(v_0(x) + \varepsilon v_1(x, \frac{x}{\varepsilon}) + \varepsilon \nabla_x \mathbf{v}_1(x, \frac{x}{\varepsilon}) + \nabla_y \mathbf{v}_1(x, \frac{x}{\varepsilon}) \right) \right] \\ & = \int_D f(x) \cdot \left[v_0(x) + \varepsilon v_1(x, \frac{x}{\varepsilon}) + \varepsilon \nabla_x \mathbf{v}_1(x, \frac{x}{\varepsilon}) + \nabla_y \mathbf{v}_1(x, \frac{x}{\varepsilon}) \right]. \end{aligned}$$

Passing to the two scale limit, we have

$$\begin{aligned} & \int_D \int_Y [a(x, y)(\operatorname{curl} u_0 + \operatorname{curl}_y u_1) \cdot (\operatorname{curl} v_0 + \operatorname{curl}_y v_1) + b(x, y)(u_0 + \nabla_y \mathbf{u}_1) \cdot (v_0 + \nabla_y \mathbf{v}_1)] dy dx \\ & = \int_D f(x) \cdot v_0(x) dx + \int_D \int_Y f(x) \cdot \nabla_y \mathbf{v}_1(x, y) dy dx = \int_D f(x) \cdot v_0(x) dx. \end{aligned} \quad (2.6)$$

Equation (2.5) follows from a density argument. From Lax-Milgram Lemma, (2.6) has a unique solution (u_0, u_1, \mathbf{u}_1) . The two scale convergence properties thus hold for the whole sequence $\{u^\varepsilon\}_\varepsilon$. \square

We now derive the cell problems and the homogenized equation.

Letting $v_0 = 0$, $v_1 = 0$, and deduce that

$$\int_D \int_Y b(x, y)(u_0 + \nabla_y \mathbf{u}_1) \cdot \nabla_y \mathbf{v}_1 dy dx = 0.$$

For each $r = 1, 2, 3$, let $w^r(x, \cdot) \in L^2(D, H_\#^1(Y)/\mathbb{R})$ be the solution of the cell problem

$$\int_D \int_Y b(x, y)(e_r + \nabla_y w^r) \cdot \nabla_y \psi dy dx = 0 \quad \forall \psi \in L^2(D, H_\#^1(Y)/\mathbb{R}) \quad (2.7)$$

where e_r is the vector in \mathbb{R}^3 with all the components being 0, except the r th component which equals 1. This is the standard cell problem in elliptic homogenization. From this we have

$$\mathbf{u}_1(x, y) = w^r(x, y) u_{0r}(x). \quad (2.8)$$

Therefore

$$\int_D \int_Y b(x, y)(u_0 + \nabla_y \mathbf{u}_1) \cdot v_0 dx dy = \int_D b^0(x) u_0(x) \cdot v_0(x) dx,$$

where the positive definite matrix $b^0(x)$ is defined as

$$b_{ij}^0(x) = \int_Y b(x, y)(e_j + \nabla w^j(x, y)) \cdot (e_i + \nabla w^i(x, y)) dy. \quad (2.9)$$

Let $v_0 = 0$ and $\mathbf{v}_1 = 0$. We have

$$\int_D \int_Y a(x, y)(\operatorname{curl} u_0 + \operatorname{curl}_y u_1) \cdot \operatorname{curl}_y v_1 dy dx = 0$$

for all $v_1 \in L^2(D, \tilde{H}_\#(\operatorname{curl}, Y))$. For each $r = 1, 2, 3$, let $N^r \in L^2(D, \tilde{H}_\#(\operatorname{curl}, Y))$ be the solution of

$$\int_D \int_Y a(x, y)(e_r + \operatorname{curl}_y N^r) \cdot \operatorname{curl}_y v dy dx = 0 \quad (2.10)$$

for all $v \in L^2(D, \tilde{H}_\#(\operatorname{curl}, Y))$. We have

$$u_1 = \operatorname{curl} u_0(x)_r N^r(x, y). \quad (2.11)$$

The homogenized coefficient a^0 is determined by

$$a_{ij}^0(x) = \int_Y a(x, y)(e_j + \operatorname{curl}_y N^j) \cdot (e_i + \operatorname{curl}_y N^i) dy. \quad (2.12)$$

We have

$$\int_D \int_Y a(x, y) (\operatorname{curl} u_0 + \operatorname{curl}_y u_1) \cdot \operatorname{curl} v_0 dx dy = \int_D a^0(x) \operatorname{curl} u_0(x) \cdot \operatorname{curl} v_0(x) dx \quad \forall v_0 \in H_0(\operatorname{curl}, D).$$

The homogenized problem is

$$\int_D [a^0(x) \operatorname{curl} u_0(x) \cdot \operatorname{curl} v_0(x) + b^0(x) u_0(x) \cdot v_0(x)] dx = \int_D f(x) \cdot v_0(x) dx \quad \forall v_0 \in H_0(\operatorname{curl}, D). \quad (2.13)$$

3 Homogenization error when $u_0 \in H^1(\operatorname{curl}, D)$

We show in this section the homogenization error when the solution u_0 of the homogenized problem (2.13) belongs to the regularity space $H^1(\operatorname{curl}, D)$. We follow the approach in [6].

Theorem 3.1 *Assume that $a(x, y) \in C(\bar{D}, C(\bar{Y}))$, $u_0 \in H^1(\operatorname{curl}; D)$, $N^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\operatorname{curl}_y N^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $w^r \in C^1(\bar{D}, C^1(\bar{Y}))$ for all $r = 1, 2, 3$, then*

$$\|u^\varepsilon - [u_0 + \nabla_y u_1(\cdot, \frac{\cdot}{\varepsilon})]\|_{L^2(D)^3} \leq c\varepsilon^{1/2}$$

and

$$\|\operatorname{curl} u^\varepsilon - [\operatorname{curl} u_0 + \operatorname{curl}_y u_1(\cdot, \frac{\cdot}{\varepsilon})]\|_{L^2(D)^3} \leq c\varepsilon^{1/2}.$$

Proof We consider the function

$$u_1^\varepsilon(x) = u_0(x) + \varepsilon N^r(x, \frac{x}{\varepsilon}) \operatorname{curl} u_0(x)_r + \varepsilon \nabla u_1(x, \frac{x}{\varepsilon}).$$

We have

$$\begin{aligned} \operatorname{curl}(a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon &= \operatorname{curl} a(x, \frac{x}{\varepsilon}) \left[\operatorname{curl} u_0(x) + \varepsilon \operatorname{curl}_x N^r(x, \frac{x}{\varepsilon}) \operatorname{curl} u_0(x)_r + \operatorname{curl}_y N^r(x, \frac{x}{\varepsilon}) \operatorname{curl} u_0(x)_r + \right. \\ &\quad \left. \varepsilon \nabla \operatorname{curl} u_0(x)_r \times N^r(x, \frac{x}{\varepsilon}) \right] + \\ &\quad + b(x, \frac{x}{\varepsilon}) \left[u_0(x) + \nabla_y w^r(x, \frac{x}{\varepsilon}) u_{0r}(x) + \varepsilon N^r(x, \frac{x}{\varepsilon}) \operatorname{curl} u_0(x)_r + \varepsilon \nabla_x w^r(x, \frac{x}{\varepsilon}) u_{0r}(x) + \varepsilon w^r(x, \frac{x}{\varepsilon}) \nabla u_{0r}(x) \right] \\ &= \operatorname{curl} a^0(x) \operatorname{curl} u_0(x) + b^0(x) u_0(x) + \operatorname{curl} G_r(x, \frac{x}{\varepsilon}) \operatorname{curl} u_0(x)_r + g_r(x, \frac{x}{\varepsilon}) u_{0r}(x) + \varepsilon \operatorname{curl} I(x) + \varepsilon J(x) \end{aligned}$$

where the vector functions $G_r(x, y)$ and $g_r(x, y)$ are defined by

$$(G_r)_i(x, y) = a_{ir}(x, y) + a_{ij}(x, y) \operatorname{curl}_y N^r(x, y)_j - a_{ir}^0(x), \quad (3.1)$$

$$(g_r)_i(x, y) = b_{ir}(x, y) + b_{ij}(x, y) \frac{\partial w^r}{\partial y_j}(x, y) - b_{ir}^0(x); \quad (3.2)$$

and

$$I(x) = a(x, \frac{x}{\varepsilon}) \left[\operatorname{curl}_x N^r(x, \frac{x}{\varepsilon}) \operatorname{curl} u_0(x)_r + \nabla \operatorname{curl} u_0(x)_r \times N^r(x, \frac{x}{\varepsilon}) \right]$$

and

$$J(x) = b(x, \frac{x}{\varepsilon}) \left[N^r(x, \frac{x}{\varepsilon}) \operatorname{curl} u_0(x)_r + \nabla_x w^r(x, \frac{x}{\varepsilon}) u_{0r}(x) + w^r(x, \frac{x}{\varepsilon}) \nabla u_{0r}(x) \right].$$

From (2.10), we have that $\operatorname{curl}_y G_r(x, y) = 0$. Further from (2.12) $\int_Y G_r(x, y) dy = 0$. We thus deduce that there is a function $\tilde{G}_r(x, y)$ such that $G_r(x, y) = \nabla_y \tilde{G}_r(x, y)$. From (2.7), we have that $\operatorname{div}_y g_r(y) = 0$

¹Indeed for Theorems 3.1 and 4.1, we only need weaker regularity conditions $N^r \in W^{1,\infty}(D, L^\infty(Y))$ and $\operatorname{curl}_y N^r \in W^{1,\infty}(D, L^\infty(Y))$, and $w^r \in W^{1,\infty}(D, W^{1,\infty}(Y))$.

and from (2.9) $\int_Y g_r(x, y) dy = 0$. Therefore there is a function \tilde{g}_r such that $g_r(x, y) = \text{curl}_y \tilde{g}_r(x, y)$. Thus for all $\phi \in \mathcal{D}(D)^3$

$$\begin{aligned} & \langle \text{curl}(a^\varepsilon \text{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \text{curl}(a^0 \text{curl} u_0) - b^0 u_0, \phi \rangle = \\ & \int_D G_r(x, \frac{x}{\varepsilon}) \text{curl} u_0(x)_r \cdot \text{curl} \phi(x) dx + \int_D g_r(x, \frac{x}{\varepsilon}) u_{0r}(x) \cdot \phi(x) dx \\ & + \varepsilon \int_D I(x) \cdot \text{curl} \phi(x) dx + \varepsilon \int_D J(x) \cdot \phi(x) dx \\ & = -\varepsilon \int_D \tilde{G}_r(x, \frac{x}{\varepsilon}) \text{div}(\text{curl} u_0(x)_r \text{curl} \phi(x)) dx - \varepsilon \int_D \nabla_x \tilde{G}_r(x, \frac{x}{\varepsilon}) \text{curl} u_0(x)_r \cdot \text{curl} \phi(x) dx \\ & + \varepsilon \int_D \tilde{g}_r(x, \frac{x}{\varepsilon}) \cdot \text{curl}(u_{0r}(x) \phi(x)) dx - \varepsilon \int_D \text{curl}_x \tilde{g}_r(x, \frac{x}{\varepsilon}) u_{0r}(x) \cdot \phi(x) dx \\ & + \varepsilon \int_D I(x) \cdot \text{curl} \phi(x) dx + \varepsilon \int_D J(x) \cdot \phi(x). \end{aligned}$$

We note that

$$\int_D \tilde{G}_r(x, \frac{x}{\varepsilon}) \text{div}(\text{curl} u_0(x)_r \text{curl} \phi(x)) dx = \int_D \tilde{G}_r(x, \frac{x}{\varepsilon}) \nabla \text{curl} u_0(x)_r \cdot \text{curl} \phi(x) dx$$

and

$$\int_D \tilde{g}_r(x, \frac{x}{\varepsilon}) \cdot \text{curl}(u_{0r}(x) \phi(x)) dx = \int_D \tilde{g}_r(x, \frac{x}{\varepsilon}) \cdot (u_{0r}(x) \text{curl} \phi(x) + \nabla u_{0r}(x) \times \phi(x)) dx.$$

As $\nabla_y \tilde{G}(x, \cdot) = G(x, \cdot) \in H^1(Y)^3$ so $\Delta_y \tilde{G}(x, \cdot) \in L^2(Y)$. Thus $\tilde{G}(x, \cdot) \in H^2(Y)$ which implies $\tilde{G}(x, \cdot) \in C(\bar{Y})$. As $G(x, \cdot) \in C^1(\bar{D}, H_\#^1(Y)^3)$, we deduce that $\tilde{G}(x, y) \in C^1(\bar{D}, H^2(Y)) \subset C^1(\bar{D}, C(\bar{Y}))$. The construction of \tilde{g}_r in Jikov et al. [6] implies that $g_r \in C^1(\bar{D}, C(\bar{Y}))$ (see Hoang and Schwab [5]). Thus

$$\left| \int_D \tilde{G}_r(x, \frac{x}{\varepsilon}) \text{div}(\text{curl} u_0(x)_r \text{curl} \phi(x)) dx \right| \leq c \|\text{curl} \phi\|_{L^2(D)^3},$$

and

$$\left| \int_D \tilde{g}_r(x, \frac{x}{\varepsilon}) \cdot \text{curl}(u_{0r}(x) \phi(x)) dx \right| \leq c(\|\text{curl} \phi\|_{L^2(D)^3} + \|\phi\|_{L^2(D)^3}).$$

As $N^r \in C^1(\bar{D}, C(\bar{Y}))$ and $w^r \in C^1(\bar{D}, C(\bar{Y}))$, $\|I\|_{L^2(D)}$ and $\|J\|_{L^2(D)}$ are uniformly bounded with respect to ε . From these we conclude that

$$|\langle \text{curl}(a^\varepsilon \text{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \text{curl}(a^0 \text{curl} u_0) - b^0 u_0, \phi \rangle| \leq c\varepsilon(\|\text{curl} \phi\|_{L^2(D)^3} + \|\phi\|_{L^2(D)^3})$$

where $\langle \cdot \rangle$ denotes the duality pairing of V and V' . Using a density argument, we have that this holds for all $\phi \in H_0(\text{curl}, D)$, thus

$$\|\text{curl}(a^\varepsilon \text{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \text{curl}(a^0 \text{curl} u_0) - b^0 u_0\|_{V'} \leq c\varepsilon$$

so

$$\|\text{curl}(a^\varepsilon \text{curl}(u_1^\varepsilon - u^\varepsilon)) + b^\varepsilon(u_1^\varepsilon - u^\varepsilon)\|_{V'} \leq c\varepsilon.$$

Let $\tau^\varepsilon(x)$ be a function in $\mathcal{D}(D)$ such that $\tau^\varepsilon(x) = 1$ outside an ε neighbourhood of ∂D and $\sup_{x \in D} |\nabla \tau^\varepsilon(x)| < c$ where c is independent of ε . Let

$$w_1^\varepsilon = u_0(x) + \varepsilon \tau^\varepsilon(x) N^r(x, \frac{x}{\varepsilon}) \text{curl} u_0(x)_r + \varepsilon \nabla[\tau^\varepsilon(x) \mathbf{u}_1(x, \frac{x}{\varepsilon})].$$

The function $w_1^\varepsilon(x)$ belongs to $H_0(\text{curl}, D)$. We note that

$$u_1^\varepsilon - w_1^\varepsilon = \varepsilon(1 - \tau^\varepsilon(x)) N^r(x, \frac{x}{\varepsilon}) \text{curl} u_0(x)_r + \varepsilon \nabla[(1 - \tau^\varepsilon(x)) \mathbf{u}_1(x, \frac{x}{\varepsilon})].$$

From this,

$$\begin{aligned} \text{curl}(u_1^\varepsilon - w_1^\varepsilon) &= \varepsilon \text{curl}_x N^r(x, \frac{x}{\varepsilon}) \text{curl} u_0(x)_r (1 - \tau^\varepsilon(x)) + \text{curl}_y N^r(x, \frac{x}{\varepsilon}) \text{curl} u_0(x)_r (1 - \tau^\varepsilon(x)) - \\ &\quad \varepsilon \text{curl} u_0(x)_r \nabla \tau^\varepsilon(x) \times N^r(x, \frac{x}{\varepsilon}) + \varepsilon(1 - \tau^\varepsilon(x)) \nabla \text{curl} u_0(x)_r \times N^r(x, \frac{x}{\varepsilon}). \end{aligned}$$

Let $D^\varepsilon \subset D$ be the ε neighbourhood of the boundary ∂D . We note that

$$\|\phi\|_{L^2(D^\varepsilon)}^2 \leq c\varepsilon^2 \|\phi\|_{H^1(D)}^2 + c\varepsilon \|\phi\|_{L^2(\partial D)}^2 \leq c\varepsilon \|\phi\|_{H^1(D)}^2$$

(see Hoang and Schwab [5]). We therefore deduce that $\|\operatorname{curl} u_0(x)_r\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{1/2}$. From these we have

$$\|\operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)\|_{L^2(D)^3} \leq c\varepsilon^{1/2}.$$

On the other hand,

$$\begin{aligned} u_1^\varepsilon - w_1^\varepsilon &= \varepsilon(1 - \tau^\varepsilon(x))N^r(x, \frac{x}{\varepsilon})\operatorname{curl} u_0(x)_r - \varepsilon\nabla\tau^\varepsilon(x)w^r(x, \frac{x}{\varepsilon})u_{0r}(x) + \\ &\varepsilon(1 - \tau^\varepsilon(x))\nabla_x w^r(x, \frac{x}{\varepsilon})u_{0r}(x) + (1 - \tau^\varepsilon(x))\nabla_y w^r(x, \frac{x}{\varepsilon})u_{0r}(x) + \varepsilon(1 - \tau^\varepsilon(x))w^r(x, \frac{x}{\varepsilon})\nabla u_{0r}(x). \end{aligned}$$

Using the fact that $N^r \in C^1(\bar{D}, C(\bar{Y}))$, $w^r \in C^1(\bar{D}, C^1(\bar{Y}))$ and $\|u_{0r}\|_{L^2(D^\varepsilon)^3} \leq c\varepsilon^{1/2}$, we deduce that $\|u_1^\varepsilon - w_1^\varepsilon\|_{L^2(D)^3} \leq c\varepsilon^{1/2}$. Therefore

$$\|\operatorname{curl}(a^\varepsilon \operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)) + b^\varepsilon(u_1^\varepsilon - w_1^\varepsilon)\|_{V'} \leq c\varepsilon^{1/2}.$$

Thus

$$\|\operatorname{curl}(a^\varepsilon \operatorname{curl}(u^\varepsilon - w_1^\varepsilon)) + b^\varepsilon(u^\varepsilon - w_1^\varepsilon)\|_{V'} \leq c\varepsilon^{1/2}.$$

Since $u^\varepsilon - w_1^\varepsilon \in H_0(\operatorname{curl}, D)$, $\|u^\varepsilon - w_1^\varepsilon\|_{H(\operatorname{curl}, D)} \leq c\varepsilon^{1/2}$. From these we have $\|u^\varepsilon - u_1^\varepsilon\|_{H(\operatorname{curl}, D)} \leq c\varepsilon^{1/2}$. We then get the conclusion. \square

4 Homogenization error when $u_0 \in H^s(\operatorname{curl}, D)$ for $0 < s < 1$

In this section, we consider the case where the solution u_0 of the homogenized problem (2.13) belongs to the weaker regularity space $H^s(\operatorname{curl}, D)$ for $0 < s < 1$. We have the following result.

Theorem 4.1 *Assume that $a \in C(\bar{D}, C(\bar{Y}))$, $u_0 \in H^s(\operatorname{curl}, D)$, $N^r \in C^1(\bar{D}, C(\bar{Y}))^3$, $\operatorname{curl}_y N^r \in C^1(\bar{D}, C(\bar{Y}))^3$, and $w^r \in C^1(\bar{D}, C^1(\bar{Y}))$ for all $r = 1, 2, 3$, then*

$$\|u^\varepsilon - [u_0 + \nabla_y u_1(\cdot, \frac{\cdot}{\varepsilon})]\|_{L^2(D)^3} \leq c\varepsilon^{s/(1+s)}$$

and

$$\|\operatorname{curl} u^\varepsilon - [\operatorname{curl} u_0 + \operatorname{curl}_y u_1(\cdot, \frac{\cdot}{\varepsilon})]\|_{L^2(D)^3} \leq c\varepsilon^{s/(1+s)}.$$

Proof We consider a set of M open cubes Q_i ($i = 1, \dots, M$) of size ε^t for $t > 0$ to be chosen later such that $D \subset \bigcup_{i=1}^M Q_i$ and $Q_i \cap D \neq \emptyset$. Each cube Q_i intersects with only a finite number, which does not depend on ε , of other cubes. We consider a partition of unity that consists of M functions ρ_i such that ρ_i has support in Q_i , $\sum_{i=1}^M \rho_i(x) = 1$ for all $x \in D$ and $|\nabla \rho_i(x)| \leq c\varepsilon^{-t}$ for all x (indeed such a set of cubes Q_i and a partition of unity can be constructed from a fixed set of cubes of size $O(1)$ by rescaling). For $r = 1, 2, 3$ and $i = 1, \dots, M$, we denote by

$$U_i^r = \frac{1}{|Q_i|} \int_{Q_i} \operatorname{curl} u_0(x)_r dx$$

and

$$V_i^r = \frac{1}{|Q_i|} \int_{Q_i} u_0(x)_r dx$$

(as $u_0 \in H^s(D)^3$ and $\operatorname{curl} u_0 \in H^s(D)^3$, for the Lipschitz domain D , we can extend each of them, separately, continuously outside D and understand u_0 and $\operatorname{curl} u_0$ as these extensions (see Wloka [10] Theorem 5.6)). Let U_i and V_i denote the vector (U_i^1, U_i^2, U_i^3) and (V_i^1, V_i^2, V_i^3) respectively. Let B be the unit cube in \mathbb{R}^3 . From Poincare inequality, we have

$$\int_B \left| \phi - \int_B \phi(x) dx \right|^2 dx \leq c \int_B |\nabla \phi(x)|^2 dx \quad \forall \phi \in H^1(B).$$

By translation and scaling, we deduce that

$$\int_{Q_i} \left| \phi - \frac{1}{|Q_i|} \int_{Q_i} \phi(x) dx \right|^2 dx \leq c\varepsilon^{2t} \int_{Q_i} |\nabla \phi(x)|^2 dx \quad \forall \phi \in H^1(Q_i)$$

i.e

$$\left\| \phi - \frac{1}{|Q_i|} \int_{Q_i} \phi(x) dx \right\|_{L^2(Q_i)} \leq c\varepsilon^t \|\phi\|_{H^1(Q_i)}.$$

Together with

$$\left\| \phi - \frac{1}{|Q_i|} \int_{Q_i} \phi(x) dx \right\|_{L^2(Q_i)} \leq c \|\phi\|_{L^2(Q_i)}$$

we deduce from interpolation that

$$\left\| \phi - \frac{1}{|Q_i|} \int_{Q_i} \phi(x) dx \right\|_{L^2(Q_i)} \leq c\varepsilon^{ts} \|\phi\|_{H^s(Q_i)} \quad \forall \phi \in H^s(Q_i).$$

Thus

$$\int_{Q_i} |\operatorname{curl} u_0(x)_r - U_i^r|^2 dx \leq c\varepsilon^{2ts} \|(\operatorname{curl} u_0)_r\|_{H^s(Q_i)}^2. \quad (4.1)$$

Let

$$u_1^\varepsilon(x) = u_0(x) + \varepsilon N^r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) + \varepsilon \nabla[w^r(x, \frac{x}{\varepsilon}) V_j^r \rho_j(x)].$$

We have

$$\begin{aligned} & \operatorname{curl}(a^\varepsilon(x) \operatorname{curl} u_1^\varepsilon(x)) + b^\varepsilon(x) u_1^\varepsilon(x) \\ &= \operatorname{curl} a(x, \frac{x}{\varepsilon}) \left[\operatorname{curl} u_0(x) + \varepsilon \operatorname{curl}_x N^r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) + \operatorname{curl}_y N^r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) \right. \\ & \quad \left. + \varepsilon (U_j^r \nabla \rho_j) \times N^r(x, \frac{x}{\varepsilon}) \right] + \\ & \quad b(x, \frac{x}{\varepsilon}) \left[u_0(x) + \varepsilon N^r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) + \varepsilon \nabla_x w^r(x, \frac{x}{\varepsilon}) V_j^r \rho_j(x) + \nabla_y w^r(x, \frac{x}{\varepsilon}) V_j^r \rho_j(x) + \right. \\ & \quad \left. \varepsilon w^r(x, \frac{x}{\varepsilon}) V_j^r \nabla \rho_j(x) \right] \\ &= \operatorname{curl}(a^0(x) \operatorname{curl} u_0(x)) + b^0(x) u_0(x) + \operatorname{curl} \left[G_r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) \right] + g_r(x, \frac{x}{\varepsilon}) V_j^r \rho_j(x) + \varepsilon \operatorname{curl} I(x) + \varepsilon J(x) + \\ & \quad \operatorname{curl} [(a^\varepsilon(x) - a^0(x)) (\operatorname{curl} u_0(x) - U_j \rho_j(x))] + (b^\varepsilon(x) - b^0(x)) (u_0(x) - V_j \rho_j(x)), \end{aligned}$$

where $G_r(x, y)$ and $g_r(x, y)$ are defined as in (3.1) and (3.2) respectively, and

$$I(x) = a(x, \frac{x}{\varepsilon}) \left[\operatorname{curl}_x N^r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) + (U_j^r \nabla \rho_j(x)) \times N^r(x, \frac{x}{\varepsilon}) \right],$$

and

$$J(x) = b(x, \frac{x}{\varepsilon}) \left[N^r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) + \nabla_x w^r(x, \frac{x}{\varepsilon}) V_j^r \rho_j(x) + w^r(x, \frac{x}{\varepsilon}) V_j^r \nabla \rho_j(x) \right].$$

Therefore for $\phi \in H_0(\operatorname{curl}, D)$

$$\begin{aligned} & \langle \operatorname{curl}(a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \operatorname{curl}(a^0 \operatorname{curl} u_0) - b^0 u_0, \phi \rangle \\ &= \int_D G_r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) \cdot \operatorname{curl} \phi dx + \int_D g_r(x, \frac{x}{\varepsilon}) (V_j^r \rho_j(x)) \cdot \phi(x) dx + \\ & \quad \varepsilon \int_D I(x) \cdot \operatorname{curl} \phi(x) dx + \varepsilon \int_D J(x) \cdot \phi(x) dx + \int_D (a^\varepsilon - a^0) (\operatorname{curl} u_0(x) - U_j \rho_j) \cdot \operatorname{curl} \phi(x) dx \\ & \quad + \int_D (b^\varepsilon - b^0) (u_0 - V_j \rho_j) \cdot \phi dx. \end{aligned}$$

We have that

$$\begin{aligned} & \int_D G_r(x, \frac{x}{\varepsilon}) (U_j^r \rho_j) \cdot \operatorname{curl} \phi dx = \int_D \left[\varepsilon \nabla \tilde{G}_r(x, \frac{x}{\varepsilon}) - \varepsilon \nabla_x \tilde{G}_r(x, \frac{x}{\varepsilon}) \right] (U_j^r \rho_j(x)) \cdot \operatorname{curl} \phi dx \\ &= -\varepsilon \int_D \tilde{G}_r(x, \frac{x}{\varepsilon}) \operatorname{div}[(U_j^r \rho_j) \operatorname{curl} \phi] dx - \varepsilon \int_D \nabla_x \tilde{G}_r(x, \frac{x}{\varepsilon}) (U_j^r \rho_j) \cdot \operatorname{curl} \phi dx. \end{aligned}$$

We note that

$$\left| \int_D \nabla_x \tilde{G}_r(x, \frac{x}{\varepsilon})(U_j^r \rho_j) \cdot \operatorname{curl} \phi dx \right| \leq c \|(U_j^r \rho_j)\|_{L^2(D)} \|\operatorname{curl} \phi\|_{L^2(D)^3}.$$

From

$$\|U_j^r \rho_j\|_{L^2(D)}^2 = \int_D (U_j^r)^2 \rho_j(x)^2 dx + \sum_{i \neq j} \int_D U_i^r U_j^r \rho_i(x) \rho_j(x) dx,$$

and the fact that the support of each function ρ_i intersects only with the support of a finite number (which does not depend on ε) of other functions ρ_j in the partition of unity, we deduce

$$\|U_j^r \rho_j\|_{L^2(D)}^2 \leq c \sum_{j=1}^M (U_j^r)^2 |Q_j| = c \sum_{j=1}^M \frac{1}{|Q_j|} \left(\int_{Q_j} \operatorname{curl} u_0(x)_r dx \right)^2 \leq c \sum_{j=1}^M \int_{Q_j} \operatorname{curl} u_0(x)_r^2 dx \leq c \int_D \operatorname{curl} u_0(x)_r^2 dx.$$

Thus

$$\left| \varepsilon \int_D \nabla_x \tilde{G}_r(x, \frac{x}{\varepsilon})(U_j^r \rho_j) \cdot \operatorname{curl} \phi dx \right| \leq c\varepsilon \|\operatorname{curl} \phi\|_{L^2(D)^3}.$$

We also have

$$\varepsilon \int_D \tilde{G}_r(x, \frac{x}{\varepsilon}) \operatorname{div}[(U_j^r \rho_j) \operatorname{curl} \phi] dx = \varepsilon \int_D \tilde{G}_r(x, \frac{x}{\varepsilon}) [(U_j^r \nabla \rho_j(x))] \cdot \operatorname{curl} \phi dx \leq c\varepsilon \|U_j^r \nabla \rho_j\|_{L^2(D)^3} \|\operatorname{curl} \phi\|_{L^2(D)^3}.$$

As the support of each function ρ_i intersects with the support of a finite number of other functions ρ_j and $\|\nabla \rho_j\|_{L^\infty(D)} \leq c\varepsilon^{-t}$, we have

$$\|U_j^r \nabla \rho_j\|_{L^2(D)^3}^2 \leq c \sum_{j=1}^M (U_j^r)^2 |Q_j| \|\nabla \rho_j\|_{L^\infty(D)}^2 \leq c\varepsilon^{-2t} \sum_{j=1}^M (U_j^r)^2 |Q_j| \leq c\varepsilon^{-2t},$$

so

$$\varepsilon \int_D \tilde{G}_r(x, \frac{x}{\varepsilon}) \operatorname{div}[(U_j^r \rho_j) \operatorname{curl} \phi] dx \leq c\varepsilon \|U_j^r \nabla \rho_j\|_{L^2(D)^3} \|\operatorname{curl} \phi\|_{L^2(D)^3} \leq c\varepsilon^{1-t} \|\operatorname{curl} \phi\|_{L^2(D)^3}.$$

We therefore deduce that

$$\left| \int_D G_r(x, \frac{x}{\varepsilon})(U_j^r \rho_j) \cdot \operatorname{curl} \phi dx \right| \leq c\varepsilon^{1-t} \|\operatorname{curl} \phi\|_{L^2(D)^3}.$$

We have

$$\int_D g_r(x, \frac{x}{\varepsilon})(V_j^r \rho_j) \cdot \phi(x) dx = \int_D [\varepsilon \operatorname{curl} \tilde{g}_r(x, \frac{x}{\varepsilon}) - \varepsilon \operatorname{curl}_x \tilde{g}_r(x, \frac{x}{\varepsilon})](V_j^r \rho_j) \cdot \phi dx.$$

Arguing similarly as above, we have

$$\left| \varepsilon \int_D \operatorname{curl}_x \tilde{g}_r(x, \frac{x}{\varepsilon})(V_j^r \rho_j) \cdot \phi dx \right| \leq c\varepsilon \|V_j^r \rho_j\|_{L^2(D)^3} \|\phi\|_{L^2(D)^3} \leq c\varepsilon \|\phi\|_{L^2(D)^3},$$

and

$$\begin{aligned} \left| \varepsilon \int_D \operatorname{curl} \tilde{g}_r(x, \frac{x}{\varepsilon})(V_j^r \rho_j) \cdot \phi dx \right| &\leq \left| \varepsilon \int_D \tilde{g}_r(x, \frac{x}{\varepsilon}) \cdot \operatorname{curl} [(V_j^r \rho_j) \phi] dx \right| \\ &\leq \left| \varepsilon \int_D \tilde{g}_r(x, \frac{x}{\varepsilon}) \cdot [(V_j^r \rho_j) \operatorname{curl} \phi + \phi \times (V_j^r \nabla \rho_j)] dx \right| \\ &\leq c (\varepsilon \|\operatorname{curl} \phi\|_{L^2(D)^3} + c\varepsilon^{1-t} \|\phi\|_{L^2(D)^3}) \left(\sum_{j=1}^M (V_j^r)^2 |Q_j| \right)^{1/2} \\ &\leq c (\varepsilon \|\operatorname{curl} \phi\|_{L^2(D)^3} + c\varepsilon^{1-t} \|\phi\|_{L^2(D)^3}). \end{aligned}$$

We note that

$$\|I\|_{L^2(D)^3} \leq c \sup_r [\|U_j^r \rho_j\|_{L^2(D)} + \|U_j^r \nabla \rho_j\|_{L^2(D)}] \leq c\varepsilon^{-t},$$

and

$$\|J\|_{L^2(D)^3} \leq c \sup_r [\|U_j^r \rho_j\|_{L^2(D)} + c\|V_j^r \rho_j\|_{L^2(D)} + c\|V_j^r \nabla \rho_j\|_{L^2(D)}] \leq c\varepsilon^{-t}.$$

We have further that

$$\langle \operatorname{curl}((a^\varepsilon - a^0)(\operatorname{curl} u_0 - U_j \rho_j)), \phi \rangle \leq c \|\operatorname{curl} u_0 - (U_j \rho_j)\|_{L^2(D)^3} \|\operatorname{curl} \phi\|_{L^2(D)^3}.$$

We note that

$$\int_D |(\operatorname{curl} u_0)_r - (U_j^r \rho_j)|^2 dx = \int_D \left| \sum_{j=1}^M ((\operatorname{curl} u_0)_r - U_j^r) \rho_j \right|^2 dx.$$

Using the support property of ρ_j , we have from (4.1)

$$\begin{aligned} \int_D |(\operatorname{curl} u_0)_r - (U_j^r \rho_j)|^2 dx &\leq c \sum_{j=1}^M \int_{Q_j} |(\operatorname{curl} u_0)_r - U_j^r|^2 dx \leq c\varepsilon^{2st} \sum_{j=1}^M \|(\operatorname{curl} u_0)_r\|_{H^s(Q_j)}^2 \\ &= c\varepsilon^{2st} \sum_{j=1}^M \left[\int_{Q_j} (\operatorname{curl} u_0)_r^2 dx + \int_{Q_j \times Q_j} \frac{(\operatorname{curl} u_0(x)_r - \operatorname{curl} u_0(x')_r)^2}{|x - x'|^{2s}} dx dx' \right] \\ &\leq c\varepsilon^{2st} \left[\|(\operatorname{curl} u_0)_r\|_{L^2(D)}^2 + \int_{D \times D} \frac{(\operatorname{curl} u_0(x)_r - \operatorname{curl} u_0(x')_r)^2}{|x - x'|^{2s}} dx dx' \right] \\ &= c\varepsilon^{2st} \|(\operatorname{curl} u_0)_r\|_{H^s(D)}^2. \end{aligned} \quad (4.2)$$

Thus

$$\langle \operatorname{curl}((a^\varepsilon - a^0)(\operatorname{curl} u_0 - U_j \rho_j)), \phi \rangle \leq c\varepsilon^{st} \|\operatorname{curl} \phi\|_{L^2(D)^3}.$$

Similarly, we have

$$\left| \int_D (b^\varepsilon - b^0)(u_0 - \sum_{j=1}^M V_j \rho_j) \cdot \phi dx \right| \leq c \left\| \sum_{j=1}^M (u_0 - V_j) \rho_j \right\|_{L^2(D)^3} \|\phi\|_{L^2(D)^3} \leq c\varepsilon^{st} \|\phi\|_{L^2(D)^3}.$$

Therefore

$$|\langle \operatorname{curl}(a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \operatorname{curl}(a^0 \operatorname{curl} u_0) - b^0 u_0, \phi \rangle| \leq c(\varepsilon^{1-t} + \varepsilon^{st}) \|\phi\|_V$$

i.e.

$$\|\operatorname{curl}(a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \operatorname{curl}(a^0 \operatorname{curl} u_0) - b^0 u_0\|_{V'} \leq c(\varepsilon^{1-t} + \varepsilon^{st}).$$

Thus

$$\|\operatorname{curl}(a^\varepsilon \operatorname{curl} u_1^\varepsilon) + b^\varepsilon u_1^\varepsilon - \operatorname{curl}(a^\varepsilon \operatorname{curl} u^\varepsilon) - b^\varepsilon u^\varepsilon\|_{V'} \leq c(\varepsilon^{1-t} + \varepsilon^{st}). \quad (4.3)$$

We choose τ^ε as in the previous section and consider the function

$$w_1^\varepsilon(x) = u_0(x) + \varepsilon \tau^\varepsilon(x) N^r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) + \varepsilon \nabla[\tau^\varepsilon(x) w^r(x, \frac{x}{\varepsilon}) V_j^r \rho_j].$$

We then have

$$u_1^\varepsilon - w_1^\varepsilon = \varepsilon(1 - \tau^\varepsilon(x)) N^r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) + \varepsilon \nabla[(1 - \tau^\varepsilon(x)) w^r(x, \frac{x}{\varepsilon}) V_j^r \rho_j]$$

and

$$\begin{aligned} \operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon) &= \varepsilon \operatorname{curl}_x N^r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) (1 - \tau^\varepsilon(x)) + \operatorname{curl}_y N^r(x, \frac{x}{\varepsilon}) U_j^r \rho_j(x) (1 - \tau^\varepsilon(x)) - \\ &\quad \varepsilon U_j^r \rho_j(x) \nabla \tau^\varepsilon(x) \times N^r(x, \frac{x}{\varepsilon}) + \varepsilon(1 - \tau^\varepsilon(x)) U_j^r \nabla \rho_j(x) \times N^r(x, \frac{x}{\varepsilon}). \end{aligned}$$

As shown above $\|U_j^r \rho_j\|_{L^2(D)}$ is uniformly bounded, so

$$\|\varepsilon \operatorname{curl}_x N^r(x, \frac{x}{\varepsilon})(U_j^r \rho_j)(1 - \tau^\varepsilon(x))\|_{L^2(D)^3} \leq c\varepsilon.$$

Let \tilde{D}^ε be the $3\varepsilon^t$ neighbourhood of ∂D . We note that $\operatorname{curl} u_0$ is extended continuously outside D . As shown in Hoang and Schwab [5], for $\phi \in H^1(\tilde{D}^\varepsilon)$

$$\|\phi\|_{L^2(\tilde{D}^\varepsilon)} \leq c\varepsilon^{t/2} \|\phi\|_{H^1(\tilde{D}^\varepsilon)}.$$

From this and

$$\|\phi\|_{L^2(\tilde{D}^\varepsilon)} \leq \|\phi\|_{L^2(\tilde{D}^\varepsilon)},$$

using interpolation we get

$$\|\phi\|_{L^2(\tilde{D}^\varepsilon)} \leq c\varepsilon^{st/2} \|\phi\|_{H^s(\tilde{D}^\varepsilon)} \leq c\varepsilon^{st/2} \|\phi\|_{H^s(D)},$$

$\forall \phi \in H^s(D)$ extended continuously outside D .

We then have

$$\begin{aligned} \|U_j^r \rho_j\|_{L^2(D^\varepsilon)}^2 &\leq c \sum_{j=1}^M \int_{Q_j \cap D^\varepsilon} (U_j^r)^2 \rho_j^2 dx \\ &\leq c \sum_{j=1}^M |Q_j \cap D^\varepsilon| \frac{1}{|Q_j|^2} \left(\int_{Q_j} (\operatorname{curl} u_0)_r dx \right)^2 \\ &\leq c \sum_{Q_j \cap D^\varepsilon \neq \emptyset} \frac{|Q_j \cap D^\varepsilon|}{|Q_j|} \int_{Q_j} (\operatorname{curl} u_0)_r^2 dx. \end{aligned}$$

As D^ε is the ε neighbourhood of ∂D and Q_j has size ε^t , $|Q_j \cap D^\varepsilon| \leq c\varepsilon^{1+(d-1)t}$ so $|Q_j \cap D^\varepsilon|/|Q_j| \leq c\varepsilon^{1-t}$. When $Q_j \cap D^\varepsilon \neq \emptyset$, $Q_j \subset \tilde{D}^\varepsilon$. Thus

$$\|U_j^r \rho_j\|_{L^2(D^\varepsilon)}^2 \leq c\varepsilon^{1-t} \|(\operatorname{curl} u_0)_r\|_{L^2(\tilde{D}^\varepsilon)}^2 \leq c\varepsilon^{1-t+st} \|\operatorname{curl} u_0\|_{H^s(D)^3}^2.$$

Therefore

$$\|\varepsilon \operatorname{curl}_y N^r(x, \frac{x}{\varepsilon})(U_j^r \rho_j)(1 - \tau^\varepsilon(x))\|_{L^2(D)^3} \leq c\varepsilon^{(1-t+st)/2}$$

and

$$\|\varepsilon(U_j^r \rho_j) \nabla \tau^\varepsilon(x) \times N^r(x, \frac{x}{\varepsilon})\|_{L^2(D)^3} \leq c\varepsilon^{(1-t+st)/2}.$$

Similarly we have

$$\begin{aligned} \|U_j^r \nabla \rho_j\|_{L^2(D^\varepsilon)^3}^2 &\leq c\varepsilon^{-2t} \sum_{Q_j \cap D^\varepsilon \neq \emptyset} |Q_j \cap D^\varepsilon| (U_j^r)^2 \leq c\varepsilon^{-2t} \sum_{Q_j \cap D^\varepsilon \neq \emptyset} \frac{|Q_j \cap D^\varepsilon|}{|Q_j|} \int_{Q_j} (\operatorname{curl} u_0)_r^2 dx \\ &\leq c\varepsilon^{-2t+1-t} \|\operatorname{curl} u_0\|_{L^2(\tilde{D}^\varepsilon)^3}^2 \leq c\varepsilon^{1-3t+st} \|\operatorname{curl} u_0\|_{H^s(D)^3}^2. \end{aligned}$$

Thus

$$\|\varepsilon(1 - \tau^\varepsilon(x))(U_j^r \nabla \rho_j) \times N^r(x, \frac{x}{\varepsilon})\|_{L^2(D)} \leq c\varepsilon^{(1-t)+(1-t+st)/2}.$$

Therefore

$$\|\operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)\|_{L^2(D)^3} \leq c(\varepsilon^{(1-t+st)/2} + \varepsilon^{(1-t)+(1-t+st)/2}).$$

We further have that

$$\begin{aligned} \varepsilon \nabla[(1 - \tau^\varepsilon(x))w^r(x, \frac{x}{\varepsilon})(V_j^r \rho_j)] &= -\varepsilon \nabla \tau^\varepsilon(x) w^r(x, \frac{x}{\varepsilon})(V_j^r \rho_j) + \varepsilon(1 - \tau^\varepsilon(x)) \nabla_x w^r(x, \frac{x}{\varepsilon})(V_j^r \rho_j) \\ &\quad + (1 - \tau^\varepsilon(x)) \nabla_y w^r(x, \frac{x}{\varepsilon})(V_j^r \rho_j) + \varepsilon(1 - \tau^\varepsilon(x)) w^r(x, \frac{x}{\varepsilon})(V_j^r \nabla \rho_j). \end{aligned}$$

Arguing as above, we deduce that

$$\|V_j^r \rho_j\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{(1-t+st)/2}, \quad \|V_j^r \nabla \rho_j\|_{L^2(D^\varepsilon)} \leq c\varepsilon^{(1-t+st)/2-t}.$$

Therefore

$$\|\varepsilon \nabla[(1 - \tau^\varepsilon(x))w^r(x, \frac{x}{\varepsilon})(V_j^r \rho_j)]\|_{L^2(D)^3} \leq c(\varepsilon^{(1-t+st)/2} + \varepsilon^{1-t+(1-t+st)/2}).$$

Thus

$$\|u_1^\varepsilon - w_1^\varepsilon\|_{L^2(D)^3} \leq c(\varepsilon^{(1-t+st)/2} + \varepsilon^{(1-t)+(1-t+st)/2}).$$

Choosing $t = 1/(s+1)$ we have

$$\|\operatorname{curl}(a^\varepsilon \operatorname{curl}(u_1^\varepsilon - w_1^\varepsilon)) + b^\varepsilon(u_1^\varepsilon - w_1^\varepsilon)\|_{V'} \leq c\varepsilon^{s/(s+1)}.$$

This together with (4.3) gives

$$\|\operatorname{curl}(a^\varepsilon \operatorname{curl}(u^\varepsilon - w_1^\varepsilon)) + b^\varepsilon(u^\varepsilon - w_1^\varepsilon)\|_{V'} \leq c\varepsilon^{s/(s+1)}.$$

Thus

$$\|u^\varepsilon - w_1^\varepsilon\|_V \leq c\varepsilon^{s/(s+1)}$$

which implies

$$\|u^\varepsilon - u_1^\varepsilon\|_V \leq c\varepsilon^{s/(s+1)}. \quad (4.4)$$

We note that

$$\operatorname{curl} u_1^\varepsilon = \operatorname{curl} u_0(x) + \operatorname{curl}_y N^r(x, \frac{x}{\varepsilon})(U_j^r \rho_j) + \varepsilon \operatorname{curl}_x N^r(x, \frac{x}{\varepsilon})(U_j^r \rho_j) + \varepsilon N^r(x, \frac{x}{\varepsilon}) \times (U_j^r \nabla \rho_j).$$

From

$$\|\varepsilon \operatorname{curl}_x N^r(x, \frac{x}{\varepsilon})(U_j^r \rho_j)\|_{L^2(D)^3} \leq c\varepsilon, \quad \text{and} \quad \|\varepsilon N^r(x, \frac{x}{\varepsilon}) \times (U_j^r \nabla \rho_j)\|_{L^2(D)^3} \leq c\varepsilon \varepsilon^{-t} = c\varepsilon^{s/(1+s)},$$

we deduce that

$$\|\operatorname{curl} u_1^\varepsilon - \operatorname{curl} u_0 - \operatorname{curl}_y N^r(x, \frac{x}{\varepsilon})(U_j^r \rho_j)\|_{L^2(D)^3} \leq c\varepsilon^{s/(s+1)}.$$

From (4.2)

$$\|\operatorname{curl} u_0 - (U_j \rho_j)\|_{L^2(D)^3} \leq c\varepsilon^{ts} = c\varepsilon^{s/(s+1)},$$

we get

$$\|\operatorname{curl} u_1^\varepsilon - [\operatorname{curl} u_0 + \operatorname{curl}_y N^r(x, \frac{x}{\varepsilon})(\operatorname{curl} u_0)_r]\|_{L^2(D)^3} \leq c\varepsilon^{s/(s+1)}.$$

This together with (4.4) implies

$$\|\operatorname{curl} u^\varepsilon - [\operatorname{curl} u_0 + \operatorname{curl}_y N^r(x, \frac{x}{\varepsilon})(\operatorname{curl} u_0)_r]\|_{L^2(D)^3} \leq c\varepsilon^{s/(s+1)}.$$

□

5 Regularity of N^r , w^r and u_0

We show in this section that the regularity requirements for the solutions $N^r(x, y)$, $w^r(x, y)$ of the cell problems (2.10) and (2.7), and of the solution u_0 of the homogenized problem (2.13) hold. We first prove the following lemma.

Lemma 5.1 *Let $\psi \in H_\#(\operatorname{curl}, Y) \cap H_\#(\operatorname{div}, Y)$. Assume further that $\int_Y \psi(y) dy = 0$. Then $\psi \in H_\#^1(Y)$ and*

$$\|\psi\|_{H^1(Y)^3} \leq c(\|\operatorname{curl}_y \psi\|_{L^2(Y)^3} + \|\operatorname{div} \psi\|_{L^2(Y)^3}).$$

Proof Let $\omega \subset \mathbb{R}^3$ be a smooth domain such that $\omega \supset Y$. Let $\eta \in \mathcal{D}(\omega)$ be such that $\eta(y) = 1$ when $y \in Y$. We have

$$\operatorname{curl}_y(\eta\psi) = \eta\operatorname{curl}_y\psi + \nabla_y\eta \times \psi \in L^2(\omega)^3$$

and

$$\operatorname{div}_y(\eta\psi) = \nabla_y\eta \cdot \psi + \eta\operatorname{div}_y\psi \in L^2(\omega)^3.$$

Together with the boundary condition, we conclude that $\eta\psi \in H^1(\omega)^3$ so $\psi \in H^1(Y)^3$.

We note that

$$\int_Y (\operatorname{div}\psi(y)^2 + |\operatorname{curl}_y\psi(y)|^2)dy = \sum_{i,j=1}^3 \int_Y \left(\frac{\partial\psi_i}{\partial y_j} \right)^2 + \sum_{i \neq j} \int_Y \frac{\partial\psi_i}{\partial y_i} \frac{\partial\psi_j}{\partial y_j} dy - \sum_{i \neq j} \int_Y \frac{\partial\psi_j}{\partial y_i} \frac{\partial\psi_i}{\partial y_j} dy.$$

Assume that ψ is a smooth periodic function. We have

$$\int_Y \frac{\partial\psi_i}{\partial y_i} \frac{\partial\psi_j}{\partial y_j} dy = \int_Y \left[\frac{\partial}{\partial y_i} \left(\psi_i \frac{\partial\psi_j}{\partial y_j} \right) - \psi_i \frac{\partial^2\psi_j}{\partial y_i \partial y_j} \right] dy = - \int_Y \psi_i \frac{\partial^2\psi_j}{\partial y_i \partial y_j} dy$$

as ψ is periodic. Similarly, we have

$$\int_Y \frac{\partial\psi_i}{\partial y_j} \frac{\partial\psi_j}{\partial y_i} dy = \int_Y \left[\frac{\partial}{\partial y_j} \left(\psi_i \frac{\partial\psi_j}{\partial y_i} \right) - \psi_i \frac{\partial^2\psi_j}{\partial y_j \partial y_i} \right] dy = - \int_Y \psi_i \frac{\partial^2\psi_j}{\partial y_j \partial y_i} dy.$$

Thus

$$\int_Y \frac{\partial\psi_i}{\partial y_i} \frac{\partial\psi_j}{\partial y_j} dy = \int_Y \frac{\partial\psi_i}{\partial y_j} \frac{\partial\psi_j}{\partial y_i} dy.$$

Therefore

$$\|\nabla_y\psi\|_{L^2(Y)^3}^2 = \|\operatorname{div}\psi\|_{L^2(Y)}^2 + \|\operatorname{curl}_y\psi\|_{L^2(Y)^3}^2.$$

Using a density argument, this holds for all $\phi \in H_{\#}^1(Y)$. As $\int_Y \psi(y)dy = 0$, from Poincare inequality we deduce

$$\|\psi\|_{H^1(Y)} \leq c(\|\operatorname{div}\psi\|_{L^2(Y)} + \|\operatorname{curl}_y\psi\|_{L^2(Y)^3}).$$

□

We then prove the following result on the regularity of N^r .

Proposition 5.2 *Assume that $a(x, y) \in C^1(\bar{D}, C^2(\bar{Y}))^{3 \times 3}$ and is isotropic, then $\operatorname{curl}_y N^r(x, y) \in C^1(\bar{D}, C(\bar{Y}))^3$ and we can choose a version of N^r in $L^2(D, \tilde{H}_{\#}(\operatorname{curl}, Y))$ so that $N^r(x, y) \in C^1(\bar{D}, C(\bar{Y}))^3$.*

Proof As a is isotropic, we understand in this proof that it is a scalar function. From (2.10), we have

$$\operatorname{curl}_y(\operatorname{curl}_y N^r)(x, \cdot) = -\frac{1}{a(x, \cdot)} \nabla_y a(x, \cdot) \times \operatorname{curl}_y N^r(x, \cdot) - \frac{1}{a(x, \cdot)} \operatorname{curl}_y(a(x, \cdot)e_r) \in L^2(Y)^3.$$

Therefore $\operatorname{curl}_y N^r \in H_{\#}(\operatorname{curl}, Y)$. This together with $\operatorname{div}_y \operatorname{curl}_y N^r = 0$ and Lemma 5.1 implies $\operatorname{curl}_y N^r \in H_{\#}^1(Y)^3$.

Fixing an index $q = 1, 2, 3$, we have

$$\operatorname{curl}_y \frac{\partial}{\partial y_q} \operatorname{curl}_y N^r = \frac{\partial}{\partial y_q} \operatorname{curl}_y \operatorname{curl}_y N^r = -\frac{\partial}{\partial y_q} \left(\frac{1}{a} \nabla_y a \times \operatorname{curl}_y N^r \right) - \frac{\partial}{\partial y_q} \left(\frac{1}{a} \operatorname{curl}_y(ae_r) \right) \in L^2(Y)^3.$$

With $\operatorname{div}_y \frac{\partial}{\partial y_q} \operatorname{curl}_y N^r = 0$, we deduce that $\frac{\partial}{\partial y_q} \operatorname{curl}_y N^r(x, \cdot) \in H^1(Y)^3$ for all $q = 1, 2, 3$ so $\operatorname{curl}_y N^r(x, \cdot) \in H^2(Y)^3 \subset C(\bar{Y})^3$.

We now show that $\operatorname{curl}_y N^r \in C^1(\bar{D}, H^2(Y))^3 \subset C^1(\bar{D}, C(\bar{Y}))^3$.

Fix $h \in \mathbb{R}^3$. From (2.10) we have

$$\begin{aligned} \operatorname{curl}_y(a(x, y)\operatorname{curl}_y(N^r(x + h, y) - N^r(x, y))) &= -\operatorname{curl}_y((a(x + h, y) - a(x, y))e_r) - \\ &\quad \operatorname{curl}_y((a(x + h, y) - a(x, y))\operatorname{curl}_y N^r(x + h, y)). \end{aligned}$$

This together with the smoothness of a and the uniformly boundedness of $\text{curl}_y N^r(x, y)$ in $L^2(Y)^3$ gives

$$\lim_{h \rightarrow 0} \|\text{curl}_y(N^r(x+h, \cdot) - N^r(x, \cdot))\|_{L^2(Y)^3} = 0. \quad (5.1)$$

We also have that

$$\begin{aligned} \text{curl}_y \text{curl}_y(N^r(x+h, y) - N^r(x, y)) &= -\frac{1}{a} \nabla_y a \times \text{curl}_y((N^r(x+h, y) - N^r(x, y))) \\ &\quad - \frac{1}{a} \text{curl}_y((a(x+h, y) - a(x, y))e_r) \\ &\quad - \frac{1}{a} \text{curl}_y((a(x+h, y) - a(x, y))\text{curl}_y N^r(x+h, y)) \end{aligned} \quad (5.2)$$

so

$$\lim_{|h| \rightarrow 0} \|\text{curl}_y \text{curl}_y(N^r(x+h, \cdot) - N^r(x, \cdot))\|_{L^2(Y)^3} = 0. \quad (5.3)$$

From Lemma 5.1, we have

$$\lim_{|h| \rightarrow 0} \|\text{curl}_y(N^r(x+h, \cdot) - N^r(x, \cdot))\|_{H^1(Y)^3} = 0. \quad (5.4)$$

From (5.2), we get

$$\lim_{|h| \rightarrow 0} \|\text{curl}_y \text{curl}_y(N^r(x+h, \cdot) - N^r(x, \cdot))\|_{H^1(Y)^3} = 0. \quad (5.5)$$

We have

$$\begin{aligned} \text{curl}_y \left[a(x, y) \text{curl}_y \left(\frac{N^r(x+h, y) - N^r(x, y)}{h} \right) \right] &= -\text{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} \right) e_r \right) - \\ &\quad \text{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} \right) \text{curl}_y N^r(x+h, y) \right). \end{aligned}$$

Let $\chi^r(x, \cdot) \in \tilde{H}_\#(\text{curl}, Y)$ be the solution of the problem

$$\text{curl}_y(a(x, y) \text{curl}_y \chi^r(x, \cdot)) = -\text{curl}_y \left(\frac{\partial a}{\partial x_q} e_r \right) - \text{curl}_y \left(\frac{\partial a}{\partial x_q} \text{curl}_y N^r(x, y) \right).$$

We deduce that

$$\begin{aligned} &\text{curl}_y \left(a(x, y) \text{curl}_y \left(\frac{N^r(x+h, y) - N^r(x, y)}{h} - \chi^r(x, y) \right) \right) \\ &= -\text{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a}{\partial x_q}(x, y) \right) e_r \right) \\ &\quad - \text{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a}{\partial x_q} \right) \text{curl}_y N^r(x+h, y) \right) \\ &\quad - \text{curl}_y \left(\frac{\partial a}{\partial x_q} (\text{curl}_y(N^r(x+h, y) - N^r(x, y))) \right). \end{aligned} \quad (5.6)$$

Let $h \in \mathbb{R}^3$ be a vector whose all components are 0 except the q th component. We have

$$\begin{aligned} \left\| \text{curl}_y \left(\frac{N^r(x+h, \cdot) - N^r(x, \cdot)}{h} - \chi^r(x, \cdot) \right) \right\|_{L^2(Y)^3} &\leq c \sup_{y \in Y} \left\| \frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a}{\partial x_q}(x, y) \right\|_{L^\infty(Y)} + \\ &\quad c \|\text{curl}_y(N^r(x+h, \cdot) - N^r(x, \cdot))\|_{L^2(Y)^3}. \end{aligned} \quad (5.7)$$

Thus from (5.1) we have

$$\lim_{|h| \rightarrow 0} \left\| \text{curl}_y \left(\frac{N^r(x+h, \cdot) - N^r(x, \cdot)}{h} - \chi^r(x, \cdot) \right) \right\|_{L^2(Y)^3} = 0. \quad (5.8)$$

From (5.6), we have

$$\begin{aligned}
& \operatorname{curl}_y \operatorname{curl}_y \left(\frac{N^r(x+h, y) - N^r(x, y)}{h} - \chi^r(x, y) \right) \\
&= -\frac{1}{a} \nabla_y a \times \operatorname{curl}_y \left(\frac{N^r(x+h, y) - N^r(x, y)}{h} - \chi^r(x, y) \right) \\
&\quad - \frac{1}{a} \operatorname{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a(x, y)}{\partial x_q} \right) e_r \right) \\
&\quad - \frac{1}{a} \operatorname{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a}{\partial x_q} \right) \operatorname{curl}_y N^r(x+h, y) \right) \\
&\quad - \frac{1}{a} \operatorname{curl}_y \left(\frac{\partial a}{\partial x_q} \operatorname{curl}_y (N^r(x+h, y) - N^r(x, y)) \right)
\end{aligned}$$

which converges to 0 in the $L^2(Y)$ norm when $|h| \rightarrow 0$ due to (5.1), (5.4) and (5.8). From Lemma 5.1, we have

$$\lim_{|h| \rightarrow 0} \left\| \operatorname{curl}_y \left(\frac{N^r(x+h, \cdot) - N^r(x, \cdot)}{h} - \chi^r(x, \cdot) \right) \right\|_{H^1(Y)^3} = 0. \quad (5.9)$$

Let $p = 1, 2, 3$. We then have

$$\begin{aligned}
& \frac{\partial}{\partial y_p} \operatorname{curl}_y \operatorname{curl}_y \left(\frac{N^r(x+h, y) - N^r(x, y)}{h} - \chi^r(x, y) \right) \\
&= -\frac{\partial}{\partial y_p} \left[\frac{1}{a} \nabla a \times \operatorname{curl}_y \left(\frac{N^r(x+h, y) - N^r(x, y)}{h} - \chi^r(x, y) \right) \right] \\
&\quad - \frac{\partial}{\partial y_p} \left[\frac{1}{a} \operatorname{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a(x, y)}{\partial x_q} \right) e_r \right) \right] \\
&\quad - \frac{\partial}{\partial y_p} \left[\frac{1}{a} \operatorname{curl}_y \left(\left(\frac{a(x+h, y) - a(x, y)}{h} - \frac{\partial a}{\partial x_q} \right) \operatorname{curl}_y N^r(x+h, y) \right) \right] \\
&\quad - \frac{\partial}{\partial y_p} \left[\frac{1}{a} \operatorname{curl}_y \left(\frac{\partial a}{\partial x_q} \operatorname{curl}_y (N^r(x+h, y) - N^r(x, y)) \right) \right]
\end{aligned}$$

which converges to 0 in $L^2(Y)$ for each x due to (5.5), (5.9) and the uniform boundedness of $\|\operatorname{curl} N^r(x, \cdot)\|_{H^2(Y)^3}$. Hence

$$\lim_{|h| \rightarrow 0} \left\| \frac{\partial}{\partial y_p} \operatorname{curl}_y \left(\frac{N^r(x+h, \cdot) - N^r(x, \cdot)}{h} - \chi^r(x, \cdot) \right) \right\|_{H^1(Y)^3} = 0$$

i.e.

$$\frac{\partial}{\partial x_q} \operatorname{curl}_y N^r(x, \cdot) = \chi^r(x, \cdot)$$

in the $H^2(Y)^3$ norm. Thus $\operatorname{curl}_y N^r \in C^1(\bar{D}, H^2(Y))^3$.

We can always choose a version of N^r such that $\operatorname{div}_y N^r = 0$. Indeed, let $\Phi(x, \cdot) \in L^2(D, H^1_\#(Y))$ be a function such that $\Delta_y \Phi = -\operatorname{div}_y N^r$, then $\operatorname{curl}_y(N^r + \nabla_y \Phi) = \operatorname{curl}_y N^r$ and $\operatorname{div}_y(N^r + \nabla_y \Phi) = 0$. Thus we assume that N^r and χ^r are chosen so that they are divergence free with respect to y . Further, we choose them so that $\int_Y N^r(x, y) dy = 0$ and $\int_Y \chi^r(x, y) dy = 0$. From (5.9) and Lemma 5.1, we have

$$\lim_{|h| \rightarrow 0} \left\| \left(\frac{N^r(x+h, \cdot) - N^r(x, \cdot)}{h} - \chi^r(x, \cdot) \right) \right\|_{H^1(Y)^3} = 0$$

We have further from (5.9) that

$$\lim_{|h| \rightarrow 0} \left\| \operatorname{curl}_y \frac{\partial}{\partial y_q} \left(\frac{N^r(x+h, \cdot) - N^r(x, \cdot)}{h} - \chi^r(x, \cdot) \right) \right\|_{L^2(Y)^3} = 0$$

so

$$\lim_{|h| \rightarrow 0} \left\| \frac{\partial}{\partial y_p} \left(\frac{N^r(x+h, \cdot) - N^r(x, \cdot)}{h} - \chi^r(x, \cdot) \right) \right\|_{H^1(Y)^3} = 0.$$

Therefore

$$\lim_{|h| \rightarrow 0} \left\| \frac{N^r(x+h, \cdot) - N^r(x, \cdot)}{h} - \chi^r(x, \cdot) \right\|_{H^2(Y)^3} = 0.$$

This shows that $N^r \in C^1(\bar{D}, H^2(Y))^3 \subset C^1(\bar{D}, C(\bar{Y}))^3$. \square

Proposition 5.3 Assume that $b(x, y) \in C^1(\bar{D}, C^2(\bar{Y}))^{3 \times 3}$. The solution w^r of cell problem (2.7) belongs to $C^1(\bar{D}, C^1(\bar{Y}))$.

Proof The cell problem (2.7) can be written as

$$-\nabla_y \cdot (b(x, y) \nabla_y w^r(x, y)) = \nabla_y (b(x, y) e_r).$$

Fixing $x \in \bar{D}$, the right hand side is bounded uniformly in $H^1(Y)$ so $w^r(x, \cdot)$ is uniformly bounded in $H^3(Y)$ from elliptic regularity (see McLean [7] Theorem 4.16). For $h \in \mathbb{R}^3$, we note that

$$\begin{aligned} -\nabla_y \cdot [b(x, y) \nabla_y (w^r(x+h, y) - w^r(x, y))] &= \nabla_y \cdot [(b(x+h, y) - b(x, y)) e_r] \\ &\quad + \nabla_y \cdot [(b(x+h, y) - b(x, y)) \nabla_y w^r(x+h, y)] := i_1. \end{aligned}$$

As $\int_Y w^r(x, y) dy = 0$, we have

$$\begin{aligned} \|w^r(x+h, y) - w^r(x, y)\|_{H^1(Y)} &\leq c \|\nabla_y (w^r(x+h, y) - w^r(x, y))\|_{L^2(Y)} \\ &\leq c \|(b(x+h, y) - b(x, y)) e_r\|_{L^2(Y)} + c \|(b(x+h, y) - b(x, y)) \nabla_y w^r(x+h, y)\|_{L^2(Y)} \end{aligned}$$

which converges to 0 when $|h| \rightarrow 0$. Fixing $x \in \bar{D}$, we then have from Theorem 4.16 of [7] that

$$\|w^r(x+h, y) - w^r(x, y)\|_{H^3(Y)} \leq \|w^r(x+h, y) - w^r(x, y)\|_{H^1(Y)} + \|i_1\|_{H^1(Y)} \quad (5.10)$$

which converges to 0 when $|h| \rightarrow 0$. Fixing an index q , let $h \in \mathbb{R}^3$ be a vector whose components are all zero except the q th component. Let $\eta(x, \cdot) \in H_{\#}^1(Y)/\mathbb{R}$ be the solution of the problem

$$-\nabla_y \cdot [b(x, y) \nabla_y \eta(x, y)] = \nabla_y \cdot \left[\frac{\partial b}{\partial x_q} e_r \right] + \nabla_y \cdot \left[\frac{\partial b}{\partial x_q} \nabla_y w^r(x, y) \right].$$

We have

$$\begin{aligned} -\nabla_y \cdot \left[b(x, y) \nabla_y \left(\frac{w^r(x+h, y) - w^r(x, y)}{h} - \eta \right) \right] &= \nabla_y \cdot \left[\left(\frac{b(x+h, y) - b(x, y)}{h} - \frac{\partial b}{\partial x_q} \right) e_r \right] \\ &\quad + \nabla_y \cdot \left[\left(\frac{b(x+h, y) - b(x, y)}{h} - \frac{\partial b}{\partial x_q} \right) \nabla_y w^r(x+h, y) \right] \\ &\quad + \nabla_y \cdot \left[\frac{\partial b(x, y)}{\partial x_q} (\nabla_y w^r(x+h, y) - \nabla_y w^r(x, y)) \right] := i_2. \end{aligned}$$

From (5.10) and the regularity of b , $\lim_{|h| \rightarrow 0} \|i_2(x, \cdot)\|_{H^1(Y)} = 0$. As $\int_Y w^r(x, y) dy = 0$ and $\int_Y \eta(x, y) dy = 0$, we have that

$$\lim_{|h| \rightarrow 0} \left\| \frac{w^r(x+h, \cdot) - w^r(x, \cdot)}{h} - \eta(x, \cdot) \right\|_{H^1(Y)} = 0.$$

Therefore from Theorem 4.16 of [7], we have

$$\left\| \frac{w^r(x+h, \cdot) - w^r(x, \cdot)}{h} - \eta(x, \cdot) \right\|_{H^3(Y)} \leq \left\| \frac{w^r(x+h, \cdot) - w^r(x, \cdot)}{h} - \eta(x, \cdot) \right\|_{H^1(Y)} + \|i_2\|_{H^1(Y)}$$

which converges to 0 when $|h| \rightarrow 0$. Thus $w^r \in C^1(\bar{D}, H^3(Y)) \subset C^1(\bar{D}, C^1(\bar{Y}))$. \square

For the regularity of the solution u_0 of the homogenized problem (2.13) we have the following result.

Proposition 5.4 Assume that D is a Lipschitz polygonal domain, and the coefficient $a(\cdot, y)$ is uniformly Lipschitz with respect to x , then there is a constant $0 < s < 1$ so that $\text{curl } u_0 \in H^s(D)$.

Proof When $a(x, y)$ is uniformly Lipschitz with respect to x , from (2.10), $\|\operatorname{curl}_y N^r(x, \cdot)\|_{L^2(Y)}$ is a Lipschitz function of x , so from (2.12) we have that a^0 is Lipschitz with respect to x . As a^0 is positive definite, $(a^0)^{-1}$ is Lipschitz. Let $U = a^0 \operatorname{curl} u_0$. We have from (2.13) that $U \in H(\operatorname{curl}; D)$, $\operatorname{div}((a^0)^{-1}U) = 0$ and $(a^0)^{-1}U \cdot n = 0$ on ∂D where n is the outward normal vector on ∂D . The conclusion follows from Lemma 4.2 of Hiptmair [4]. \square

Remark 5.5 *If a^0 is isotropic, we have from (2.13) that*

$$\operatorname{curl} \operatorname{curl} u_0 = -\frac{1}{a^0} \nabla a^0 \times \operatorname{curl} u_0 - \frac{1}{a^0} b^0 u_0 + \frac{1}{a^0} f \in L^2(D)$$

so $u_0 \in H^1(\operatorname{curl}, D)$. However, even if a is isotropic, a^0 may not be isotropic.

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